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THE GLOBAL STABILITY OF THE STEADY ROTATIONS OF A SOLID†

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The dynamical Euler equations describing the motion of a non-symmetrical solid about the centre of mass in the field of a constant external moment and a dissipative one are considered. It is assumed that the external moment specified with respect to axes attached to the body acts about the intermediate central axis of inertia of the body. The conditions for global asymptotic stability as well as the stability in total of steady rotations of the solid are obtained.

1. STATEMENT OF THE PROBLEM

CONSIDER the Euler equations of motion of a solid about the centre of mass written in a system of coordinates attached to the body [1, 2]

$$I_1\dot{\omega}_1 = -(I_3 - I_2)\ \omega_2\omega_3 - k_1 I_1\omega_1 + F_1(1\ 2\ 3) \tag{1.1}$$

Here ω_1, ω_2 and ω_3 are the projections of the vector of the instantaneous angular velocity ω on to the coordinate axes, $I_1 > I_2 > I_3$ are the principle central moments of inertia of the body, k_1, k_3 and k_3 are the damping factors along the corresponding axes [3], and F_1, F_2 and F_3 are the components of a perturbing moment **F**.

Equations (1.1) have been examined by many researchers. In particular, Greenhill proved that system (1.1) has an exact solution in elliptic functions [4] at $\mathbf{F} = 0$ and $k_1 = k_2 = k_3$. Under the same assumptions, an expansion of the solutions of this system in terms of the small parameter $\epsilon = (I_1 - I_2)/I_3$ was constructed in [5]. The asymptotic properties of solutions when there are small dissipative and small steady external moments was examined [2] by the averaging method. The small-scale stability of the rotation of a body was examined in [3] by the direct Lyapunov method assuming that one component of the vector **F** is non-zero.

Note that if we replace $-\omega \times \mathbf{L}$ by $\omega \times \mathbf{L}$ in the system of equations (1.1) where $\mathbf{L} = I\omega$, $I = \operatorname{diag}(I_1, I_2, I_3)$, we obtain equations well known in fluid dynamics that describe the motion of a liquid gyroscope. Those equations have been examined, for instance, in [6–8]‡ for the case when the friction is isotropic ($k_1 = k_2 = k_3$).

In this paper, Eqs (1.1) are examined when $F_1 = F_3 = 0$ and when $F_2 = F_0 > 0$. This means, physically, that the external moment is orientated along the intermediate axis of inertia of the body. The main object of this paper is to single out the domains in the space of the parameters of system (1.1) for which the set of steady rotations of the body is stable for any perturbations of the vector of the instantaneous angular velocity ω. The investigation is non-local in nature and is carried out using the direct Lyapunov method and the results of the qualitative theory of multidimensional dynamical systems [6, 9, 10].

Let us introduce the new parameters and time

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‡See also MOROZOV A. V., Global stability of dynamical systems with a non-unique position of equilibrium. Candidate dissertation, Leningrad, 1989.

$$p_0 = 2\frac{I_2 - I_3}{I_1 - I_3} \in [0, 2], \quad f_0 = 2\frac{\sqrt{I_1 I_3}}{I_1 - I_3} F_0, \quad l_i = 2\frac{\sqrt{I_1 I_2 I_3}}{I_1 - I_3} k_i, \quad i = 1, 2, 3$$

$$t' = t\frac{I_1 - I_3}{2\sqrt{I_1 I_2 I_3}}$$

By changing the variables

$$\sqrt{I_2}\omega_2 = x + f_0 l_2^{-1}, \quad \sqrt{2I_1}\,\omega_1 = -y, \quad \sqrt{2I_3}\,\omega_3 = z$$

we then reduce Eqs (1.1) to the form

$$\begin{aligned} x^{\cdot} &= -l_2 x + yz, \quad y^{\cdot} = -l_1 y - p_0 f_0 l_2^{-1} z - p_0 xz \\ z^{\cdot} &= -l_3 z - (2 - p_0) f_0 l_2^{-1} y - (2 - p_0) xy \end{aligned} \tag{1.2}$$

We see that the cases of dynamical symmetrical bodies for which $I_2 = I_3$ and $I_1 = I_2$, correspond to the values $p_0 = 0$ and $p_0 = 2$, respectively. These cases have been examined in detail and will not be considered here [11]. Also note that equality $I_2 = (I_1 + I_3)/2$ corresponds to the value $p_0 = 1$

When the condition

$$f_0 < f_* \quad (f_* = l_2\sqrt{l_1 l_3}/\sqrt{p_0(2 - p_0)}) \tag{1.3}$$

holds, the system of Eqs (1.2) has a unique asymptotically stable (in Lyapunov's sense) equilibrium position C_0 with coordinates $x = 0$, $y = 0$, $z = 0$. It loses its stability when $f_0 > f_*$, and transfers it to two produced equilibrium positions $C_{1,2}$ with coordinates

$$x_{1,2} = \frac{f_* - f_0}{l_2}, \quad y_{1,2} = \pm\left[\frac{p_0 f_0 f_*}{l_1 l_2} - \frac{l_2 l_3}{2 - p_0}\right]^{1/2}, \quad z_{1,2} = \mp\left[\frac{(2 - p_0) f_0 f_*}{l_2 l_3} - \frac{l_1 l_2}{p_0}\right]^{1/2}$$

The eigenvalues of the matrix of the vector field of system (1.2) linearized at C_0 then have the form

$$\lambda_1 = -l_2, \quad \lambda_{2,3} = \tfrac{1}{2}\{-(l_1 + l_3) \pm [(l_1 - l_3)^2 + 4 l_1 l_3 f_0^2/f_*^2]^{1/2}\}$$

Note that the steady rotation of the body about the intermediate principal central axis of inertia corresponds to the equilibrium position C_0, the axis of the instantaneous rotation being collinear with the vector **F**; the other steady rotations correspond to the equilibrium positions $C_{1,2}$.

We will say that the system of equations (1.2) is globally asymptotically stable if any of its trajectory $x = x(t)$, $y = y(t)$, $z = z(t)$ approaches one of the equilibrium positions as $t \to \infty$; it is asymptotically stable in total if it has one asymptotically stable (in Lyapunov's sense) equilibrium position and is globally asymptotically stable; (1.2) is dissipative, in Levinson's sense [12], if a number $R > 0$ exists such that $\overline{\lim}\,|X(t)_{t\to\infty}| < R$ for any $X_0 = X_0(0)$. Here $X(t) = \|x(t), y(t), z(t)\|$, and the symbol $|X(t)|$ denotes the Euclidean norm of the vector. Also let us recall that a set M of a phase space is called invariant if it consists entirely of trajectories of the system.

2. THE STABILITY IN TOTAL OF THE STEADY MOTIONS OF A SOLID

Let us take into consideration the Lyapunov function of the form

$$\begin{aligned} &W(x, y, z) = p_0(2 - p_0)x^2 + \tfrac{1}{2}[(2 - p_0)y^2 + p_0 z^2] + \Theta x \\ &\Theta \in [\Theta_-, \Theta_+], \quad \Theta_\pm = 2\{p_0(2 - p_0) f_0/l_2 \pm [p_0(2 - p_0)(l_1 - \lambda)(l_3 - \lambda)]^{1/2}\} \\ &\lambda \in (0, \lambda_*), \quad \lambda_* = \min\{l_1, l_2, l_3\} \end{aligned}$$

and the non-negative number

$$\Gamma = \frac{(l_2 - 2\lambda)^2\Theta^2}{16 p_0(2 - p_0)\lambda(l_2 - \lambda)}$$

Lemma 1 (on dissipation in Levinson's sense). For any trajectory of system (1.2) the inequality

$$\overline{\lim}_{t\to+\infty} W(x(t), y(t), z(t)) \leqslant \Gamma$$

holds.

The proof of the lemma follows from the inequality [7]

$$W^{\cdot} + 2\lambda W \leqslant 2\lambda\Gamma \tag{2.1}$$

which is easily verified. Here $W^{\cdot}$ is, as usual, the derivative of the function W with respect to time, which is calculated using Eqs (1.2).

Theorem 1. If inequality (1.3) holds, the system of equations (1.2) is asymptotically stable in total.

Proof. We put $f_0 = \alpha f_*$ where α is an arbitrary number in the range (0, 1). The equation

$$\alpha\sqrt{l_1 l_3} = [(l_1 - \lambda)(l_3 - \lambda)]^{1/2}$$

has the unique root λ_0 in the interval $(0, \lambda_*)$ for which $\Theta_- = 0$. If we put $\Theta = \Theta_- = 0$, we obtain $\Gamma = 0$. From this result and from inequality (2.1) it follows that $W^{\cdot} \leqslant -2\lambda_0 W$, ($W(x, y, z)$ being a positive definite quadratic form. Note that all the conditions of the Barbashin–Krasovskii theorem on the stability in total are satisfied and this fact proves the statement of Theorem 1.

Note that when examining the problem of the stability of the steady rotations of a solid about an intermediate axis of inertia [3], inequality (1.3) was treated as the condition for asymptotic stability in Lyapunov's sense, i.e. of the stability with respect to small perturbations of the values ω_1, ω_2 and ω_3. Theorem 1 states that, when inequality (1.3) holds, the specified rotation of the body is asymptotically stable to any perturbations of values ω_1, ω_2 and ω_3.

From Theorem 1 it also follows that the domain of the stability in total of system (1.2) increases without limit in the space of the parameters f_0, l_1, l_2, l_3 for values of the parameter p_0 close to 0 or to 2.

3. THE GLOBAL STABILITY OF THE STEADY MOTIONS OF A SOLID

We will now establish the conditions for the global asymptotic stability of the system of equations (1.2). Henceforth assuming that $f_0 > f_*$ everywhere we will introduce into consideration the functions

$$V_{1,2}(x, y, z) = \sqrt{2 - p_0}\, y \pm \sqrt{p_0}\, z$$

and the following notation: W^s denotes the two-dimensional stable separatrix surface of the saddle point C_0 and $W^u = W_2^u \cup \{C_0\} \cup W_1^u$, where W_2^u and W_1^u are the one-dimensional unstable separatrices of the saddle point C_0 which go into the octants $\{x<0, y<0, z>0\}$ and $\{x<0, y>0, z<0\}$, respectively.

The main result of this part of the paper is the proof of the following assertion.

Theorem 2. If $l_1 = l_3$ then system (1.2) is globally asymptotically stable for any

$$f_0 > l_1 l_2 / \sqrt{p_0(2 - p_0)}$$

i.e. any of its trajectory approaches a certain equilibrium position as $t \to \infty$.

The proof of the theorem is based on the following simple assertions.

Lemma 2. The invariant sets correspond to the equations $V_1(x, y, z) = 0$ and $V_2(x, y, z) = 0$ in the phase space of system (1.2), and the inclusions

$$W^u \subset \{ x, y, z \mid V_1(x, y, z) = 0 \}$$

$$W^s \subset \{ x, y, z \mid V_2(x, y, z) = 0 \}$$

hold.

The proof of the lemma follows from the obvious equalities

$$V_1^{\cdot} = 0 \text{ on the set } \{x, y, z \mid V_1 = 0\},$$
$$V_2^{\cdot} = 0 \text{ on the set } \{x, y, z \mid V_2 = 0\}$$

and from the fact that the plane tightened on the eigenvectors β^1 and β^2 which correspond to the negative eigenvalues λ_1 and λ_2 coincides with the $V_2 = 0$ plane, and the eigenvector β^2 corresponding to the positive eigenvalues λ_2 lies in the $V_1 = 0$ plane (β^1, β^2 and β^3 are the eigenvectors of the matrix of the vector field of system (1.2) linearized at the point C_0).

Lemma 3. The system of equations (1.2) is asymptotically stable in total in the invariant $V_2 = 0$ plane.

Proof. Let us put

$$\sqrt{2-p_0}\, y = \sqrt{p_0}\, z$$

in (1.2).

The statement of the lemma follows directly from a consideration of the function $V(x, y) = p_0 x^2 + y^2$, for which the inequality $v^{\cdot}[x(t), y(t)] < 0$ holds, and from the Barbashin–Krasovskii theorem on the stability in total.

Lemma 4. The system of equations (1.2) is globally asymptotically stable in the invariant $V_1 = 0$ plane.

Proof. If we put

$$\sqrt{2-p_0}\, y = -\sqrt{p_0}\, z$$

in (1.2) we obtain equations that are identical, at $p_0 = 1$ and $l_1 = l_2$, apart from linear substitution, with the well-known equations of the one-dimensional flow of Burgers' fluid [6, 13]

$$\begin{gathered} x^{\cdot} = P(x, y), \quad y^{\cdot} = Q(x, y) \\ P(x, y) = -l_2 x - \sqrt{p_0^{-1}(2-p_0)}\, y^2 \\ Q(x, y) = -l_1 y + \sqrt{p_0(2-p_0)}(f_0 l_2^{-1} y + xy) \end{gathered} \tag{3.1}$$

System (3.1) is dissipative (Lemma 1), symmetrical about the x axis, and the set $\{y = 0\}$ is invariant and consists of the trajectories $x = 0$, $y = 0$; $y = 0$, $x = \gamma\exp(-l_2\epsilon)$ and $y = 0$, $x = -\gamma\exp(-l_2 t)$ ($\gamma = \text{const}$). This system has no limit cycles according to Dulac's criterion [4]. In fact, let us put, for instance, $B(x, y) = 1/y$, then the expression

$$\frac{\partial}{\partial x}[P(x, y)B(x, y)] + \frac{\partial}{\partial y}[Q(x, y)B(x, y)] = -\frac{l_2}{y}$$

keeps the sign in each of the half-planes $y > 0$ and $y < 0$. From this and from the asymptotic Lyapunov stability of the equilibrium positions C_1 and C_2 the statement of Lemma 4 follows according to Bendixson's theorem [14].

Proof of Theorem 2. Let us denote by Λ, $D = \{X \mid W(X) \leq \Gamma\}$ and Ω, respectively, the set of equilibrium positions, the domain of dissipation and the ω-limiting set [9] of system (1.2); the number Γ, the vector X and the function $W(X)$ have been specified above. From Lemma 1 it then follows that the set Ω is non-empty and $\Omega \subset D$.

Consider the function

$$V(X) = V_1(X)\, V_2(X) = (2-p_0) y^2 - p_0 z^2$$

for which

$$V^{\cdot} = -2 l_1 V \tag{3.2}$$

Let $X = X(t, X_0)$ be an arbitrary trajectory of system (1.2) and $X(0, X_0) = 0$. From (3.2) the equality

$$V(X(t,X_0)) = V(X_0)\exp(-2l_1 t), \quad t \geqslant 0 \tag{3.3}$$

then follows.

If $V(X_0) = 0$, then, by Lemmas 2–4, we have

$$X(t, X_0) \to C_i \in \Lambda \quad \text{as} \quad t \to +\infty$$

Now suppose that $V(X_0) \neq 0$, i.e. the point X_0 does not belong to the invariant set $\{X \mid V(X) = 0\}$. From (3.3) we then have

$$V(X(t, X_0)) \to 0 \quad \text{as} \quad t \to +\infty$$

Hence

$$\Omega \subset D \cap \{X \mid V(X) = 0\}$$

It is well known that Ω is the closed set consisting of the whole trajectories. But at the intersection $D \cap \{X \mid V(X) = 0\}$ of the whole trajectories there are only the equilibrium positions C_1, C_2 and C_3 and the unstable separatrices W_1^u, W_2^u of the saddle point (because of the absence of closed contours (Lemma 4) and because Bendixson's theorem [14] holds). The point C_0 is a saddle point and, hence, $\lim X(t, X_0) \neq C_0$ as $t \to \infty$. Let the point $p \in W_2^u$ be the ω-limiting point for the trajectory $X(t, X_0)$. Consider the trajectory $X(t, p)$. It is obvious that $\lim X(t, p) = C_2$ at $t \to +\infty$ and $\lim X(t, p) = C_2$ as $t \to -\infty$. Let the point q be $X(t, p)$ in the domain of attraction S_2 of the equilibrium position C_2. It is well known that q is the ω-limiting point for the trajectory $X(t, X_0)$.

Let a sequence of instants $\{t_k\}$ be such that $X(t_k, X_0) \to q$ as $k \to \infty$. Hence a value of k exists such that the point $X(t_k, X_0)$ is located in the domain S_2, and this means that $X(t, X_0) \to C_2$ as $t \to +\infty$. The latter contradicts the fact that p is the ω-limiting point for $X(t, X_0)$. Thus, $\Omega = \Lambda$. The theorem is proved.

Note that Lyapunov's function, used to prove Theorem 2, is not of constant sign and has the form

$$V(\omega_1, \omega_2, \omega_3) = I_1(I_1 - I_2)\,\omega_1^2 - I_3(I_2 - I_3)\,\omega_3^2$$

in terms of the variables ω_1, ω_2 and ω_3, and is the bundle of the first integrals of Euler's equations.

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